EPYMT TDG Group 2 Tutorial 5

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1 Curve curvature

Intuitive idea of curvature: The rate of the "bending" of the curve away from its normal.

Intuitive idea of torsion: The rate of the "bending" of the curve away from its "tangent plane".

Recall that

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

Example on computation: (Just computational exhaustive):

Example 1.1. *Find the curvature of Logarithmic spiral:* $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$*.* $a > 0, b > 0$*.*

Solution. *From the above, we have computed*

$$\alpha'(t) = (abe^{bt}\cos t - ae^{bt}\sin t, abe^{bt}\sin t + ae^{bt}\cos t).$$

and

$$\|\alpha'(t)\| = ae^{bt}\sqrt{1+b^2}.$$

Hence

$$\begin{aligned}\alpha''(t) &= (ab^2e^{bt}\cos t - abe^{bt}\sin t - abe^{bt}\sin t - ae^{bt}\cos t, ab^2e^{bt}\sin t + abe^{bt}\cos t + abe^{bt}\cos t - ae^{bt}\sin t)\\ &= (ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t, ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t)\end{aligned}$$

Therefore, we have

$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ abe^{bt}\cos t - ae^{bt}\sin t & abe^{bt}\sin t + ae^{bt}\cos t & 0 \\ ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t & ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t & 0 \end{vmatrix}$$

$$= [(abe^{bt}\cos t - ae^{bt}\sin t)(ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t)$$

$$- (abe^{bt}\sin t + ae^{bt}\cos t)(ab^2e^{bt}\cos t - 2abe^{bt}\sin t - ae^{bt}\cos t)]\mathbf{k}$$

$$= a^2e^{2bt}[b^3\sin t\cos t + 2b^2\cos^2 t - b\sin t\cos t - b^2\sin^2 t - 2b\sin t\cos t + \sin^2 t$$

$$- b^3\sin t\cos t + 2b^2\sin^2 t - b\sin t\cos t + b^2\cos^2 t + 2b\sin t\cos t + \cos^2 t]\mathbf{k}$$

$$= a^2e^{2bt}[b^2\cos^2 t + b^2a^2\sin^2 t + +\sin^2 t + \cos^2 t]\mathbf{k}$$

$$= a^2(1+b^2)e^{2bt}\mathbf{k}$$

Therefore we have

$$\begin{aligned}\kappa(t) &= \frac{\|\alpha'(t)\times\alpha''(t)\|}{\|\alpha'(t)\|^3} \\ &= \frac{a^2(1+b^2)e^{2bt}}{(ae^{bt}\sqrt{1+b^2})^3} \\ &= \frac{a^2(1+b^2)e^{2bt}}{a^3e^{3bt}(1+b^2)^{\frac{3}{2}}} \\ &= \frac{1}{ae^{bt}\sqrt{1+b^2}}\end{aligned}$$

Exercise: Prove that the curvature of the curve defined by $r = r(\theta)$ in polar coordinates is given by

$$\kappa(\theta) = \frac{|2r'^2 - rr'' + r^2|}{(r^2 + r'^2)^{\frac{3}{2}}}$$

Key:

i Recall the formula $\kappa(\theta) = \dfrac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}}$ for 2-D curve. What we need to do is to express the curve in terms of Catesterian coordinate first.

ii Note that for $r = r(\theta)$, the radial component r is a function of θ, the angular component (This is just the same as using f to express $f(x)$. Don't regard r in our question as a constant!).
Hence we can express the curve as $(r(\theta)\cos\theta, r(\theta)\sin\theta)$.

iii Then we have

$$x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta$$

$$y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta$$

$$x''(\theta) = r''(\theta)\cos\theta - r'(\theta)\sin\theta - r'(\theta)\sin\theta - r(\theta)\cos\theta = r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta$$

$$y''(\theta) = r''(\theta)\sin\theta + r'(\theta)\cos\theta + r'(\theta)\cos\theta - r(\theta)\sin\theta = r''(\theta)\sin\theta + +2r'(\theta)\cos\theta - r(\theta)\sin\theta$$

Then we have

$$
\begin{aligned}
[x'^2 + y'^2]^{\frac{3}{2}} &= [(r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2]^{\frac{3}{2}} \\
&= [r'(\theta)^2\cos^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\sin^2\theta + r'(\theta)^2\sin^2\theta + 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\cos^2\theta]^{\frac{3}{2}} \\
&= [r'(\theta)^2(\cos^2\theta + \sin^2\theta) + r(\theta)^2(\cos^2\theta + \sin^2\theta)]^{\frac{3}{2}} \\
&= (r(\theta)^2 + r'(\theta)^2)^{\frac{3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
x'y'' &= [r'(\theta)\cos\theta - r(\theta)\sin\theta][r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta] \\
&= r'(\theta)r''(\theta)\sin\theta\cos\theta + 2r'(\theta)^2\cos^2\theta - r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)r''(\theta)\sin^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta + r(\theta)^2\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
x''y' &= [r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta][r'(\theta)\sin\theta + r(\theta)\cos\theta] \\
&= r'(\theta)r''(\theta)\sin\theta\cos\theta + r(\theta)r''(\theta)\cos^2\theta - 2r'^2(\theta)\sin^2\theta - 2r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)r'(\theta)\sin\theta\cos\theta - r(\theta)^2\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
x'y'' - x''y' &= 2r'(\theta)^2[\cos^2\theta + \sin^2\theta] - r(\theta)r''(\theta)[\sin^2\theta + \cos^2\theta] + r(\theta)^2[\cos^2\theta + \sin^2\theta] \\
&= 2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\kappa(\theta) &= \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}} \\
&= \frac{|2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2|}{(r(\theta)^2 + r'(\theta)^2)^{\frac{3}{2}}}
\end{aligned}
$$

Exercise:

i Consider the curve C given by the graph of the function $y = \ln\csc x$, $0 < x < \pi$, in rectangular coordinates.

(1) Show that $\mathbf{r}(s) = (2\tan^{-1} e^s, \ln\cosh s)$, $s \in \mathbb{R}$ is an arc length parametrization of C.

(2) Show that the curvature of the curve is

$$\kappa(s) = \frac{1}{\cosh s}$$

2 Frenet frame (Part II)

Aim: Use $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ to understand a curve.

Remark: Note that $\|x\|$ is differentiable except $x = 0$.

Therefore, for a differentiable function f such that $f(t) \neq 0$ for all t, we have $\|f\|$ to be differentiable for all t.

Example 2.1. *Let $\mathbf{r}(s) : \mathbb{R} \to \mathbb{R}^3$ be a regular space curve with arc length parametrization, $\mathbf{T}(s)$ and $\mathbf{N}(s)$ be the unit tangent vector and unit normal vector respectively. Suppose $\kappa(s) > 0$ for any $s \in \mathbb{R}$ and there exists a constant c and a constant unit vector $\mathbf{u}$ such that $\langle \mathbf{T}(s), \mathbf{u} \rangle = c$ for all $s \in \mathbb{R}$.*

i *Show that $\mathbf{N}(s)$ and $\mathbf{u}$ are orthogonal for all s.*

ii *Using (a), show that there exists a constant θ such that*

$$\mathbf{u} = \cos\theta \mathbf{T}(s) + \sin\theta \mathbf{B}(s)$$

for all $s \in \mathbb{R}$.

iii *Using (b) and the Frenet formulas, or otherwise, prove that*

$$\frac{\tau(s)}{\kappa(s)} = \cot\theta.$$

Solution. *i*

$$\begin{aligned}
\langle \mathbf{T}(s), \mathbf{u} \rangle &= c \\
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{u} \rangle &= 0 \\
\langle \kappa(s)\mathbf{N}(s), \mathbf{u} \rangle &= 0 \\
\langle \mathbf{N}(s), \mathbf{u} \rangle &= 0 \textit{ (As } \kappa(s) > 0 \textit{ for any } s \in \mathbb{R} \textit{ by assumption.)}
\end{aligned}$$

Hence $\mathbf{N}(s)$ and $\mathbf{u}$ are orthogonal for all $s \in \mathbb{R}$.

ii *Note that from lecture notes, for all $s \in \mathbb{R}$, $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ forms an orthonormal basis. Then for each $s \in \mathbb{R}$ we can find $\alpha(s), \beta(s), \gamma(s) \in \mathbb{R}$, such that*

$$\mathbf{u} = \alpha(s)\mathbf{T}(s) + \beta(s)\mathbf{N}(s) + \gamma(s)\mathbf{B}(s).$$

*Here as $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is changing for all $s \in \mathbb{R}^3$, **we can't simply assume those coefficients** $\alpha(s), \beta(s), \gamma(s) \in \mathbb{R}$ **are constant for all** $s \in \mathbb{R}$. We need to further show that*

(1) $\beta(s) = 0$ *for all $s \in \mathbb{R}$.*

(2) $\alpha(s)^2 + \gamma(s)^2 = 1$ *for all $s \in \mathbb{R}$.*

(3) $\alpha(s)$, $\gamma(s)$ *is constant for all $s \in \mathbb{R}$, then we can define the constant to be α, γ respectively. Also, we need to show that*

Proof.

(1) From (a), $\langle \mathbf{N}, \mathbf{u} \rangle = \mathbf{0}$ for all $s \in \mathbb{R}$.

Hence by taking inner product with $\mathbf{N(s)}$ on both side of $\mathbf{u} = \alpha(s)\mathbf{T}(s) + \beta(s)\mathbf{N}(s) + \gamma(s)\mathbf{B}(s)$, we have

$$\begin{aligned}
\langle \mathbf{N}(s), \mathbf{u} \rangle &= \langle \mathbf{N}(s), \alpha(s)\mathbf{T}(s) + \beta(s)\mathbf{N}(s) + \gamma(s)\mathbf{B}(s) \rangle \\
0 &= \beta(s)(1) \text{ (As } \langle \mathbf{N}(s), \mathbf{T}(s) \rangle = \langle \mathbf{N}(s), \mathbf{B}(s) \rangle = 0 \text{ and } \langle \mathbf{N}(s), \mathbf{N}(s) \rangle = 1) \\
\beta(s) &= 0
\end{aligned}$$

This means we have

$$\mathbf{u} = \alpha(s)\mathbf{T}(s) + \gamma(s)\mathbf{B}(s).$$

(2) Note that from the question, $\mathbf{u}$ is an unit vector, hence we have

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{u} \rangle &= 1 \\
\langle \alpha(s)\mathbf{T}(s) + \gamma(s)\mathbf{B}, \alpha(s)\mathbf{T}(s) + \gamma(s)\mathbf{B} \rangle &= 1 \\
\alpha(s)^2(1) + \gamma(s)^2(1) &= 1 \text{ (As } \langle \mathbf{N}(s), \mathbf{T}(s) \rangle = 0 \text{ and } \langle \mathbf{T}(s), \mathbf{T}(s) \rangle = \langle \mathbf{N}(s), \mathbf{N}(s) \rangle = 1) \\
\alpha(s)^2 + \gamma(s)^2 &= 1
\end{aligned}$$

(3) Note that as $\langle \mathbf{T}(s), \mathbf{u} \rangle = c$ for all $s \in \mathbb{R}$, by taking inner product between $\mathbf{u} = \alpha(s)\mathbf{T}(s) + \gamma(s)\mathbf{B}(s)$ and $\mathbf{T}(s)$, we have

$$\begin{gathered}
\langle \mathbf{T}(s), \mathbf{u} \rangle = \langle \mathbf{T}(s), \alpha(s)\mathbf{T}(s) + \gamma(s)\mathbf{B}(s) \rangle \\
c = \alpha(s) \quad \text{for all } s \in \mathbb{R} \\
\text{Thereby } \gamma(s)^2 = 1 - \alpha(s)^2 = 1 - c^2 \text{ , which is a constant.}
\end{gathered}$$

So we can denote the coeffecient $\alpha(s)$ and $\gamma(s)$ to be constants α and γ respectively.
(You may wonder that there are two possible values of γ, namely $\sqrt{1-c^2}$ and $-\sqrt{1-c^2}$. In fact, by continuity of $\mathbf{T}, \mathbf{N}, \mathbf{B}$, there is only one choice of γ only. I will not talk about the reason here.)
Also, note that from $\alpha(s)^2 + \gamma(s)^2 = 1$, we have $|\alpha|, |\gamma| \leq 1$. So we can safely let $\alpha = \cos\theta$ and $\gamma = \sin\theta$ for some constant $\theta \in \mathbb{R}$. Then we can solve the value of θ by $\frac{\gamma}{\alpha} = \tan\theta$ if $\alpha \neq 0$. If $\alpha = 0$, just simply check the values of γ to determine whether $\theta = 0$ or π.

$\square$

iii *Differentiating the equation* $\mathbf{u} = \cos\theta\mathbf{T}(s) + \sin\theta\mathbf{B}(s)$ *with respect to both side by* s*, we have*

$$\begin{aligned}
\frac{d}{ds}\mathbf{u} &= \frac{d}{ds}[\cos\theta\mathbf{T}(s) + \sin\theta\mathbf{B}(s)] \\
\mathbf{0} &= \cos\theta\kappa(s)\mathbf{N}(s) - \sin\theta\tau(s)\mathbf{N}(s) \\
\cos\theta\kappa(s)\mathbf{N}(s) &= \sin\theta\tau(s)\mathbf{N}(s) \text{ for all } s \in \mathbb{R} \\
\cos\theta\kappa(s) &= \sin\theta\tau(s) \\
\frac{\tau(s)}{\kappa(s)} &= \cot\theta
\end{aligned}$$

Example 2.2. *Suppose all tangent of a parameterized curve $\alpha : I \to \mathbb{R}^3$ pass through a fixed point, show that the trace $\alpha(I)$ of the curve is contained in a straight line.*

Solution. *Without loss of generality, we can assume the curve to be parameterized by arc-length.*
Denote the fixed point that all tangent of a parameterized curve $\alpha : I \to \mathbb{R}^3$ pass through to be $\mathbf{p_0}$.
To show that the trace $\alpha(I)$ of the curve is contained in a straight line, we need to show that curvature $\kappa(t) = 0$ for all $t \in I$.
Then note that

i *$T(t) = \alpha'(t)$ as α is parameterized by arc-length.*

ii *$T(t) = c(t)(\alpha(t) - \mathbf{p_0})$, where $c(t) = \dfrac{1}{\|\alpha(t) - \mathbf{p_0}\|}$ as all tangent of a parameterized curve $\alpha : I \to \mathbb{R}^3$ pass through a fixed point.*
The function $c(t) = \dfrac{1}{\|\alpha(t) - \mathbf{p_0}\|}$ is used to normalize the R.H.S to make magnitude of R.H.S to be unit 1. It is differentiable as α is regular, $\|T'(t)\| \neq 0$ for all $t \in I$

From the above, we have

$$\alpha'(t) = c(t)(\alpha(t) - \mathbf{p_0})$$
$$\alpha''(t) = c'(t)(\alpha(t) - \mathbf{p_0}) + c(t)\alpha'(t)$$

Then we have

$$\begin{aligned}
\kappa(t) &= \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} \\
&= \frac{\|\alpha'(t) \times [c'(t)(\alpha(t) - \mathbf{p_0}) + c(t)\alpha'(t)]\|}{\|\alpha(t)'\|^3} \\
&= \frac{\|\alpha'(t) \times [c'(t)(\alpha(t) - \mathbf{p_0})]\|}{\|\alpha'(t)\|^3} \quad \textit{(As } \mathbf{u} \times \mathbf{u} = \mathbf{0}\textit{)} \\
&= \frac{0}{\|\alpha'(t)\|^3} \quad \textit{(As } \alpha'(t) = c(t)(\alpha(t) - \mathbf{p_0})\textit{, we have } \alpha'(t) \textit{ parallel to } (\alpha(t) - \mathbf{p_0})\textit{, therefore } \alpha'(t) \times (\alpha(t) - \mathbf{p_0}) = \mathbf{0}\textit{)} \\
&= 0
\end{aligned}$$

Hence the trace $\alpha(I)$ of the curve is contained in a straight line.

Example 2.3. *Suppose all normal of a parameterized curve $\alpha : I \to \mathbb{R}^3$ pass through a fixed point, show that the trace $\alpha(I)$ of the curve is contained in a circle.*

Solution. *Denote $\mathbf{p_0}$ be the fixed point the all normal of $\alpha : I \to \mathbb{R}^3$ pass through. To think of the centre of the "circle", one of reasonable choices is $\mathbf{p_0}$ To show that the trace $\alpha(I)$ of the curve is contained in a circle, we need to show the following two things:*

i *For all $t \in I$, $\|\alpha(t) - \mathbf{p_0}\|$ is a constant.*

ii *$\alpha(I)$ lies on a plane (i.e. Torsion $\tau(t)$=0 for all $t \in I$.)*
(Circle is a 2D object. If we do not ensure that $\alpha(I)$ lies on a plane so that αI, $\alpha(I)$ may lie on a sphere.)

Let's show the above two are correct. Without loss of generality, we can assume the curve to be parameterized by arc-length (Then we can use $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ *freely).*

i *Showing for all* $t \in I$, $\|\alpha(t) - \mathbf{p_0}\|$ *is a constant.*

Note that "$\|\alpha(t) - \mathbf{p_0}\|$ *is a constant." is equivalent to* $\|\alpha(t) - \mathbf{p_0}\|^2$ *is a constant.*

Also, differentiating $\|\alpha(t) - \mathbf{p_0}\|^2 =< \alpha(t) - \mathbf{p_0}, \alpha(t) - \mathbf{p_0} >$ *is much easier than differentiating* $\|\alpha(t) - \mathbf{p_0}\| =< \alpha(t) - \mathbf{p_0}, \alpha(t) - \mathbf{p_0} >^{\frac{1}{2}}$. *(Of course we don't like handling differentiation with square root.) Then*

$$\begin{aligned}
\frac{d}{dt}\|\alpha(t) - \mathbf{p_0}\|^2 &= \frac{d}{dt} < \alpha(t) - \mathbf{p_0}, \alpha(t) - \mathbf{p_0} > \\
&= 2 < \alpha'(t), \alpha(t) - \mathbf{p_0} > \\
&= 2 < \mathbf{T(t)}, \alpha(\mathbf{t}) - \mathbf{p_0} > \\
&= 0 \text{ (As normal passes through } \alpha(t) \text{ and } -\mathbf{p_0}\ \alpha(t) - \mathbf{p_0} \text{ is parallel to } \mathbf{N(t)}. \\
&\text{Hence } 2 < \mathbf{T(t)}, \alpha(\mathbf{t}) - \mathbf{p_0} >= \mathbf{2k} < \mathbf{T(t)}, \mathbf{N(t)} >= \mathbf{0} \text{ for some } k \in \mathbb{R})
\end{aligned}$$

ii *Showing* $\tau(t) = 0$ *for all* $t \in I$*:*

Method 1:

Recall that

$$\begin{cases}
\mathbf{T}' = \kappa\mathbf{N} \\
\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B} \\
\mathbf{B}' = -\tau\mathbf{N}
\end{cases}$$

Recall that

$$\tau = \frac{< \alpha' \times \alpha'', \alpha''' >}{\|\alpha' \times \alpha''\|^2}.$$

Then as we assume the path is parameterized by arc-length, we have $T = \alpha'$. *Then we have*

To show that $\tau(t) = 0$, *we just need to show that* $< \alpha' \times \alpha'', \alpha''' >\equiv 0$.

Thinking process: First

(1)

$$\begin{aligned}
\mathbf{T}' &= \kappa\mathbf{N} \\
(\alpha')' &= \kappa\mathbf{N} \\
\alpha'' &= \kappa\mathbf{N}
\end{aligned}$$

(2) *Also, from the question, and* $\mathbf{N}$, $\alpha - \mathbf{p_0} \neq \mathbf{0}$ *we can find some differentiable function* f *such that*

$$\mathbf{N} = f[\alpha - \mathbf{p_0}].$$

(3) *Hence, we have*

$$\alpha'' = \kappa f[\alpha - \mathbf{p_0}]$$
$$\alpha''' = \kappa' f[\alpha - \mathbf{p_0}] + \kappa f'[\alpha - \mathbf{p_0}] + \kappa f \alpha'$$

(4) *Therefore, as* $[\alpha - \mathbf{p_0}]$ *is parallel to* α''*, we have* $< \kappa' f[\alpha - \mathbf{p_0}] + \kappa f'[\alpha - \mathbf{p_0}], \alpha''' >= 0$.

Also, we have

$$< \alpha' \times \alpha'', \kappa f \alpha' >= \kappa f < \alpha' \times \alpha'', \alpha' >= 0.$$

Therefore, we have

$$< \alpha' \times \alpha'', \alpha''' >\equiv 0.$$

Method 2:

Note that $\alpha - \mathbf{p_0}$ *is parallel to* $\mathbf{N}$ *and* $\mathbf{N} = \kappa \mathbf{T}' = \kappa T'' = \kappa \alpha''$ *is parallel to* α''.

(Note that as we assume every normal passes through $\mathbf{p_0}$*, the normal must not be* $\mathbf{0}$*, hence we have* $\alpha'' \neq \mathbf{0}$ *and* $\kappa \neq 0$.*)*

Hence we have

$$(\alpha - \mathbf{p_0}) \times \alpha'' = \mathbf{0}$$
$$[(\alpha - \mathbf{p_0}) \times \alpha'']' = \mathbf{0}$$
$$\alpha' \times \alpha'' + (\alpha - \mathbf{p_0}) \times \alpha''' = \mathbf{0}$$
$$\alpha' \times \alpha'' = -(\alpha - \mathbf{p_0}) \times \alpha'''$$

Hence we have

$$\begin{aligned}\tau &= \frac{< \alpha' \times \alpha'', \alpha''' >}{\|\alpha' \times \alpha''\|^2} \\ &= \frac{< -(\alpha - \mathbf{p_0}) \times \alpha''', \alpha''' >}{\|\alpha' \times \alpha''\|^2} \\ &= 0\end{aligned}$$

This means that $\alpha(I)$ *lies on a plane.*

Hence combining both parts, we have $\alpha(I)$ *is contained in a circle.*

3 Multi-variable differentiation

3.1 Partial Derivative

Key: Treating the another variable as “constant” during partial differentiation.

Example 3.1. *Let* $f(x, y) = 3x^2y + \sin xy + e^{y^2}$.

Then for $\frac{\partial f}{\partial x}$*, we have*

$\frac{\partial f}{\partial x}(3x^2y) = 6xy$ *(As* $\frac{d}{dx}(3x^2 \cdot k) = 6kx$ *when* k *is a constant. Here we regard* y *as "constant" and take* $k = y$*.*
$\frac{\partial f}{\partial x}(\sin xy) = y \cos xy$ *(Note that* $\frac{d}{dx}(\sin kx) = k \cos kx$ *when* k *is a constant. Here we regard* y *as "constant" and take* $k = y$*.)*
$\frac{\partial f}{\partial x}(e^{y^2}) = 0$ *(As derivative of a constant function is 0 and we regard* y *as "constant", thereby* e^{y^2} *is also a "constant")*

Thereby, we have

$$\frac{\partial f}{\partial x} = 6xy + y \cos xy.$$

Similarly, we have

$$\frac{\partial f}{\partial y} = 3x^2 + x \cos xy.$$

Order: d from left to right. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$
If a function f is C^2 , then

$$f_{xy} = f_{yx}$$

If the function is not C^2, symmetry does not hold.

(Refer to `https://en.wikipedia.org/wiki/Symmetry_of_second_derivatives#Requirement_of_continuity` for more details.)

Exercise 3.1. Compute f_x, f_y, f_{xx}, f_{yy} and f_{xy} of the following functions:

i $\quad f(x,y) = x^3y - \cos(x^2+2y)$.

ii $\quad f(x,y) = \sin(e^x y) + x^3y^2 + 7x - 4y^4$.

iii $\quad f(x,y) = \ln(x^2+y^2)$.

Solution.

Answer:

i

$$\begin{aligned}
f_x &= 3x^2y + 2x\sin(x^2+2y) \\
f_y &= x^3 + 2\sin(x^2+2y) \\
f_{xx} &= 6xy + 2\sin(x^2+2y) + 4x^2\cos(x^2+2y) \\
f_{yy} &= 4\cos(x^2+2y) \\
f_{xy} &= 3x^2 + 4x\cos(x^2+2y)
\end{aligned}$$

ii

$$\begin{aligned}
f_x &= ye^x\cos(e^x y) + 3x^2y^2 + 7 \\
f_y &= e^x\cos(e^x y) + 2x^3y - 16y^3 \\
f_{xx} &= ye^x\cos(e^x y) - y^2e^{2x}\sin(e^x y) + 6xy^2 \\
f_{yy} &= -e^{2x}\sin(e^x y) + 2x^3 - 48y^2 \\
f_{xy} &= e^x\cos(e^x y) - ye^{2x}\sin(e^x y) + 6x^2y
\end{aligned}$$

3.2 Taylor series

You might have heard that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + ...$$

A natural question arise: Can we use polynomial series to calculate the value of every function?

Let's guess a formula for Taylor series: Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then note that if we differentiate both side k times, we have

$$f^k(x) = \sum_{n=k}^{\infty} (n)(n-1)...(n-k-1)a_n x^{n-k} = (k!)a_k + \sum_{n=k+1}^{\infty} (n)(n-1)...(n-k-1)a_n x^{n-k}.$$

(Note that the first k terms in the original series vanish as $\frac{d^k}{dx^k}(x^n) = 0$ for all $n = 0, 1, ..., k-1$.)

Substituting $x = 0$, we have

$$f^k(0) = k!a_k \text{ (Note that } 0^a = 0 \text{ for all positive integer } a.)$$
$$a_k = \frac{f^{(k)}(0)}{k!}$$

Thereby, we have the general formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^n.$$

Let's go back to the problem: Can we use polynomial series to calculate the value of every function? The answer is no. Firstly, from our method of finding the series, we require f to be $k-$times differentiable for all non-negative integers k. So for functions that are not infinitely many differentiable, we cannot expressed it in terms of Taylor series.

Even though the function is infinitely many differentiable, the function may still not be able to be expressed in terms of Taylor series. For instance, $f(x) = e^{-\frac{1}{x^2}}$. You may try to show that $f^{(n)}(0) = 0$ for all non-negative integers n. Then, the Taylor series of $f(x) = e^{-\frac{1}{x^2}}$ would be 0, which is impossible.

There are some question remains, but we will not address here as it reqruies understanding on Mathematical analysis. You will find more information when you take undergraduate courses:

i Does the infinite sum converge? For what value(s) of x does the infinite sum converge?

ii For an infinitely differentiable function, when can we assure that the function can be expressed in terms of its Taylor series?

3.3 Chain rule

In 1D, the chain rule is

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

(Challenge: Prove the 1D chain rule. Make sure that your proof works for all types of differentiable function)

How about for functions with more than 1 variable?

3.4 Change of variable theorem

Example 3.2 (A circle/sphere/cylinder)**.** *i* *Circle of radius r centred at (a, b):* $\{(x, y) : (x - a)^2 + (y - b)^2 = r^2\}$.

ii *Sphere of radius r centred at (a, b, c):* $\{(x, y, z) : (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2\}$.

iii *Cylinder with radius r:* $\{(x, y, z) : (x - a)^2 + (y - b)^2 + = r^2,\ a \leq z \leq b\}$.

It is computational exhaustive to set up bounds for range of integral and hard to understand the picture. Therefore, we want to use another coordinate system to understand the picture.

Definition 3.1 (Jacobian)**.** *The Jacobian of a two-variable function* $f(x,y) = (f_1(x,y), f_2(x,y))$ *is defined as*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

While the Jacobian of a three-variable function $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ *is defined as*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

Idea of determinant: To show the "change" of signed volume of a region through a linear transformation(Recall that every matrix can be regarded as a representation of a linear transformation.)

Theorem 3.1 (Change of Variable Theorem in 1D)**.**

Let $g : [a,b] \to \mathbb{R}$ *be a* C^1 *function(i.e.* g' *is continuous on* $[a,b]$*), and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous, then*

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f \circ g(t) \cdot g'(t)\ dt.$$

In general, we use the formula in the following way: Let $g : [a,b] \to \mathbb{R}$ be a C^1 **injective** function on $[a,b]$ (hence $g^{-1} : g([a,b]) \to [a,b]$ exists) (or sometimes we assume $g' \neq 0$ on $[a,b]$, which implies injectivity, but injectivity does not imply $g' \neq 0$ on $[a,b]$. For instance, $f(x) = x^3$ is injective, but $f'(0) = 0$.) and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\int_c^d f(x)\ dx = \int_{g^{-1}(c)}^{g^{-1}(d)} f \circ g(t) \cdot g'(t)\ dt.$$

Note that without injectivity, we cannot define g^{-1}, and the lower and upper limit of the integral in R.H.S.
(So now you can understand why when doing $\int_a^b \frac{1}{\sqrt{1-x^2}}\ dx$ and letting $x = \sin\theta$ (thereby $dx = \cos\theta d\theta$), we set the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This ensures injectivity of $\sin\theta$ in this domain and thereby we can well-define $\arcsin a$ and $\arcsin b$, and the integral in R.H.S: $\int_{\arcsin a}^{\arcsin b} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\ d\theta$.)

The formula is similar for higher dimension:

Theorem 3.2 (Change of Variable Theorem in higher dimension)**.**

Let $g : A \to B$ *be a diffeomorphism between two bounded open subsets* $A, B \subset \mathbb{R}^n$ *with measure zero boundary (i.e.* g *is bijective and* g, g^{-1} *are differentiable). For any continuous function* $f : B \to \mathbb{R}$ *is continuous, we have*

$$\int_B f\ dV = \int_A f \circ g(t) \cdot |\det(Dg)|\ dV.$$

where Dg *is the Jacobian of* g.

Remark. Remark: Don't be afraid of the words "diffeomorphism", "open subsets", "measure zero boundary". In our homework/tests/exams, if a question requires you to use this theorem, you don't need to check these things. You don't need to know what they mean in this course. Just simply use it. In fact, some of these concepts are from advanced Math courses!

Let's try this formula with a simple example.

Example 3.3. *Find the area of the region R bounded by a parallelogram with three of vertexes to be $(0,0),(1,1)$ and $(1,2)$. Define $\mathbf{a}=(1,1)$ and $\mathbf{b}=(1,3)$. From secondary school stuff, we know that*

$$\begin{aligned} Area &= \|\mathbf{u}\times\mathbf{v}\| \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} \\ &= |2\mathbf{k}| \\ &= 2 \end{aligned}$$

We can also use change of variable theorem to find the area: Note that the region R can be parametrized by the following: (As shown in the figure): $R=\{s(1,1)+t(1,3) : 0\le t,s\le 1\}$.
Then we can define a linear transformation from $[0,1]\times[0,1]$ to R:

$$(x,y)=(s+t,s+3t)=(f_1(s,t),f_2(s,t)).$$

Then we have

$$\begin{aligned} \frac{\partial f_1}{\partial s} &= 1 \\ \frac{\partial f_1}{\partial t} &= 1 \\ \frac{\partial f_2}{\partial s} &= 1 \\ \frac{\partial f_2}{\partial t} &= 3 \end{aligned}$$

Therefore,

$$Jacobian\ of\ f = \begin{pmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

and

$$\begin{aligned} dA &= dxdy \\ &= \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} dsdt \\ &= 2\, dsdt \end{aligned}$$

Hence by Theorem 3.2, we have

$$\begin{aligned}
Area &= \int_R 1 \; dA \\
&= \int_0^1 \int_0^1 2 \; dsdt \\
&= 2\int_0^1 [s]_0^1 dt \\
&= 2\int_0^1 dt \\
&= 2[t]_0^1 \\
&= 2
\end{aligned}$$

Now, we are going to give the formula for polar, cylindrical and spherical coordinates and find the Jacobian matrix of the transformation.

Example 3.4. *Polar coordinate:* $(x, y) = (r\cos\theta, r\sin\theta)$*, where* r *and* θ *are called radial and angular component respectively and* $0 \leq \theta < 2\pi$*.*

Then let $x = f_1(r, \theta) = r\cos\theta$ *and* $y = f_2(r, \theta) = r\sin\theta$*. We have*

$$\begin{aligned}
\frac{\partial f_1}{\partial r} &= \cos\theta \\
\frac{\partial f_1}{\partial \theta} &= -r\sin\theta \\
\frac{\partial f_2}{\partial r} &= \sin\theta \\
\frac{\partial f_2}{\partial \theta} &= r\cos\theta
\end{aligned}$$

Therefore, we have

$$Jacobian = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and

$$\begin{aligned}
dA &= dxdy \\
&= \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| drd\theta \\
&= |(r\cos^2\theta + r\sin^2\theta)| drd\theta \\
&= rdrd\theta
\end{aligned}$$

Example 3.5. *Cylindrical coordinate:* $(x, y, z) = (r\cos\theta, r\sin\theta, z)$*. This time, we have* z *as a free variable independent of* r *and* θ*. Also* $0 \leq \theta < 2\pi$

For Jacobian, similarly, let $(x, y, z) = (f_1(r, \theta, z), f_2(r, \theta, z), f_3(r, \theta, z)) = (r\cos\theta, r\sin\theta, z)$ *we have*

$$Jacobian = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} dV &= dxdydz \\ &= \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| drd\theta dz \\ &= rdrd\theta dz \end{aligned}$$

Example 3.6. *Spherical coordinate:* $(x, y, z) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$*, where* $0 \le \theta < 2\pi$ *and* $0 \le \phi < \pi$*.*
θ *can still be regarded as the angular component in the circle parallel to x-y plane. But this time,* r *does not just lie on* $x-y$ *plane only. Instead, its orientation is controlled by* ϕ*. When* $\phi = 0$*, the point is* $(r\sin 0\cos\theta, r\sin 0\sin\theta, r\cos 0) = (0, 0, r)$*, the highest point in the sphere. When* $\phi = \pi$*, the point is* $(r\sin\pi\cos\theta, r\sin\pi\sin\theta, r\cos\pi) = (0, 0, -r)$*, the lowest point in the sphere.*

For Jacobian, similarly, let $(x, y, z) = (f_1(r, \theta, \phi), f_2(r, \theta, \phi), f_3(r, \theta, z)) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$*, we have*

$$Jacobian = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \phi} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

and

$$\begin{aligned} dV &= dxdydz \\ &= \left| \det \begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix} \right| drd\theta d\phi \\ &= |\cos\phi(-r^2\sin\phi\cos\phi\sin^2\theta - r^2\sin\phi\cos\phi\cos^2\theta) - r\sin\phi(r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta)|drd\theta d\phi \\ &= |-r^2\sin\phi\cos^2\phi[\sin^2\theta + \cos^2\theta] - r^2\sin^3\phi[\cos^2\theta + \sin^2\theta]|drd\theta d\phi \\ &= |-r^2\sin\phi\cos^2\phi - r^2\sin^3\phi|drd\theta d\phi \\ &= r^2|\sin\phi||\cos^2\phi + \sin^2\phi|drd\theta d\phi \\ &= r^2|\sin\phi|drd\theta d\phi \\ &= r^2\sin\phi drd\theta d\phi \text{ (As we assume } \phi \in [0, \pi) \text{ and } 0 \le \sin\phi \le 1 \text{ for } \phi \in [0, \pi)) \end{aligned}$$

Definition 3.2 (Volume of 3D region)**.**

The volume of a 3D region R is defined to be

$$\text{Volume} = \iiint_R 1 \; dV$$

Example 3.7. *Show that volume of a sphere of radius r is $\frac{4\pi r^2}{3}$ respectively.*

Example 3.8. *Find the volume of the region R bounded by the curve $z = x^2 + y^2$ and $z = 4$.*

Step 1: *Parameterize the region for computing volume: Note that $x^2 + y^2$ can be regarded as a cirlce, so we can use polar coordinate to parametrize (x, y) as $(r\cos\theta, r\sin\theta)$, where $0 \leq \theta < 2\pi$.*

For the height z, note that it is bounded above by 4. For the lower bound, from the picture below, we can find that $z \geq x^2 + y^2 = r^2$.

Therefore, we have $r^2 \leq z \leq 4$. In particular, $r^2 \leq 4$ and thereby $r \leq 2$. So we can use cylindrical coordinate to parametrize the region and the region is $\{(r\cos\theta, r\sin\theta, z) : \; 0 \leq r \leq 2, \; 0 \leq \theta < 2\pi, \; r^2 \leq z \leq 4\}$

Step 2: *Compute the integral:*

$$\begin{aligned}
\text{Volume} &= \int_0^2 \int_0^{2\pi} \int_{r^2}^4 r \; dz d\theta dr \text{ (As the range of } z \text{ is affected by } r\text{, } dz \text{ should be put in the most inside one.)} \\
&= \int_0^2 \int_0^{2\pi} r[z]_{r^2}^4 d\theta dr \\
&= \int_0^2 \int_0^{2\pi} r(4 - r^2) d\theta dr \\
&= \int_0^2 r(4 - r^2)[\theta]_0^{2\pi} dr \\
&= 2\pi \int_0^2 4r - r^3 dr \\
&= 2\pi[2r^2 - \frac{r^4}{4}]_0^2 \\
&= 2\pi[2(2)^2 - \frac{2^4}{4}] \\
&= 2\pi(4) \\
&= 8\pi
\end{aligned}$$